Comments on Linear Transformation (Defined in Linear Algebra)

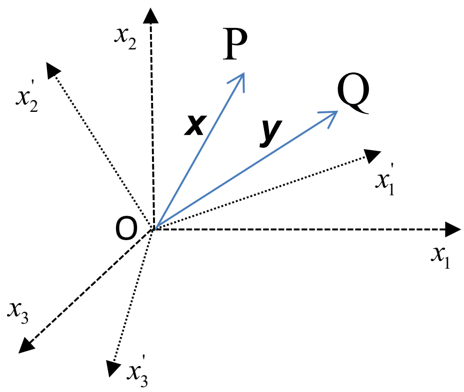

Figure: Coordinate system $Ox_1^{'}x_2^{'}x_3^{'}$ rotated with respect to $Ox_1x_2x_3$.

The two vectors $\boldsymbol{x} = \boldsymbol{OP}$ and $\boldsymbol{y} = \boldsymbol{OQ}$ are indicated as (X), (Y), respectively, with respect to the coordinate system $Ox_1x_2x_3$. These two vectors are related by the following linear transformation,

$$(Y) = [A](X) \qquad \text{(a)}$$

where $[A]$ is a 3x3 matrix with constant coefficients. On the new, rotated system $Ox_1^{'}x_2^{'}x_3^{'}$ this same linear transformation is expressed as,

$$(Y') = [A'](X') \qquad \text{(b)}$$

The relationships of the two vectors in the two coordinate systems is,

$$(X') = [C](X) \quad \text{or} \quad (X) = [C]^{-1}(X') \qquad \text{(c)}$$

$$(Y') = [C](Y) \quad \text{or} \quad (Y) = [C]^{-1}(Y') \qquad \text{(d)}$$

Here $[C]$ is an orthogonal matrix with its coefficients being the direction cosines that satisfy the orthogonality conditions, $[C][C]^{-1} = [C][C]^{T}$.

Next, we introduce the two vectors (c) and (d) in (a),

$$[C]^{-1}(Y') = [A][C]^{-1}(X') \;\Rightarrow\; (Y') = [C][A][C]^{-1}(X').$$

Comparing this last relation with (b) we can conclude that,

$$[A'] = [C][A][C]^{-1} = [C][A][C]^{T}. \qquad \text{(e)}$$

When we impose the condition that the vector $(Y) = \lambda(X)$, i.e. the two vectors should be collinear, we have from (a),

$$[A](X) = \lambda(X) \Rightarrow \{[A] - \lambda[I]\}(X) = 0. \qquad \text{(f)}$$

Where $[I]$ is the identity matrix. To obtain the solution of the last equation, we need to satisfy the condition,

$$\det\left([A]-\lambda[I]\right)=0 \tag{g}$$

The last relation is the characteristic equation of the well-known eigenvalue problem. The roots of the cubic equation are the eigenvalues, or the elements of the diagonal matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

For each eigenvalue, the corresponding eigenvector is obtained from the system of equations (f).